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# An Asymptotic Formula for Some Wavelet Series*

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We obtain an asymptotic formula describing the pointwise behavior of certain wavelet series at a neighborhood of a point of divergence. 

## 1. INTRODUCTION

Information about the order of magnitude of the Fourier coefficients of a function $f$ is not sufficient for making conclusions about the size or regularity of $f$. For example, in [3] it is shown how to obtain a continuous function from an arbitrary square-integrable function by modifying slightly the moduli of the Fourier coefficients of the latter and changing the phases in an appropriate manner.

A common principle in the study of the local regularity of a function $f$ by means of its wavelet series expansion

$$f(x)=\sum_{j,k} c_{j,k}\psi(2^{j}x-k) \tag{1}$$

is that it is enough to consider only those coefficients $c_{j,k}$ such that $\psi(2^{j}x-k)$ is localized at the neighborhood in question. For example in [2], Jaffard investigates the problem of characterizing local Hölder continuity in terms of its wavelet coefficients, while in [4, p. 116], a necessary condition for differentiability at a given point is presented in terms of the decay of the periodic wavelet coefficients.

In this note, we shall investigate the pointwise behavior of certain "sub-series" of (1), all of whose terms are localized at a given point $x_0$, and such

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that the corresponding coefficients are relatively large. Our main results yield wavelet series analogous to the classical formula

$$\sum_{n=1}^{\infty} n^{-\beta} \cos nx \simeq x^{\beta-1} \Gamma(1-\beta) \sin(\beta\pi/2), \qquad x \to +0$$

where $0 < \beta < 1$. See for example [6, p. 186]. Here, and in what follows,

$$A(x) \simeq B(x), \qquad x \to 0$$

will mean that $A(x)(B(x))^{-1} \to 1$ as $x \to 0$.

The following example is a consequence of Theorem 2 in Section 3. For $0 < \alpha \leqslant 1$, let

$$F_\alpha(x) = \sum_{j=1}^{\infty} j^{-\alpha} \psi(2^{n_j} x - k_j), \qquad x \in R \tag{2}$$

Assuming that $\psi(\theta^*) \neq 0$, we have

$$F_1(x_0 + \delta) \simeq \psi(\theta^*) \log\log(|\delta|^{-1}), \qquad \delta \to 0$$

while for $0 < \alpha < 1$,

$$F_\alpha(x_0 + \delta) \simeq C_\alpha \psi(\theta^*)(\log(|\delta|^{-1}))^{1-\alpha}, \qquad \delta \to 0,$$

where $C_\alpha = (1-\alpha)^{-1} (q^*/\log 2)^{1-\alpha}$, and $q^* = \lim jn_j^{-1}$. In obtaining this results, we have assumed that

(A) $\{k_j\}_{j=1}^{\infty}$ is any sequence of integers while $\{n_j\}_{j=1}^{\infty}$ is an increasing sequence of positive integers relatively dense in $\mathbf{N}$ in the sense that for some $M \in \mathbf{N}$,

$$\{l+1, ..., l+M\} \cap \{n_1, n_2, ...\} \neq \phi \tag{3}$$

for every integer $l \geqslant 0$,

(B) there exists $x_0 \in \mathbf{R}$ for which the sequence

$$\theta_j := 2^{n_j} x_0 - k_j, \qquad j = 1, 2, 3, ...,$$

converges to some real number $\theta^*$ (one may take, for instance, $x_0 = 0$ and $k_j = 0$ for all $j$, so that $\theta^* = 0$),

(C) the function $\psi: \mathbf{R} \to \mathbf{R}$ has a bounded derivative and that for some positive constants $C$, $N$,

$$|\psi(x)| \leqslant C(1+|x|^N)^{-1} \tag{4}$$

for all real $x$,

(D) the sequence $\{jn_j^{-1}\}$ converges.

We remark that under condition (B) above, it can likely happen that the wavelet series

$$F(x) = \sum_{j=1}^{\infty} c_j \psi(2^{n_j} x - k_j) \tag{5}$$

will be divergent at $x = x_0$ if the coefficients $c_j$'s do not tend to zero fast enough.

It was shown in [5] that under conditions (A), (B), and (C), there exists a constant $K_1 < \infty$ such that the function $F$ defined in (5) satisfies

$$|F(x_0+\delta)| \leqslant K_1 \log(|\delta|^{-1})$$

whenever $0 < |\delta| < 1/2$, provided that the sequence $\{c_j\}$ of coefficients is convergent. If in addition (D) is satisfied, then

$$\lim_{\delta \to 0} \frac{F(x_0+\delta)}{\log(|\delta|^{-1})} = \lambda_0 \psi(\theta^*) \tag{6}$$

where

$$\lambda_0 = (\log 2)^{-1} \lim_{j \to \infty} \frac{jc_j}{n_j}.$$

## 2. THE MAIN ESTIMATE

If the sequence $\{c_j\}$ of coefficients tends to zero, (6) does not tell us much about the pointwise asymptotic behavior of $F(x)$ near $x_0$. Our main objective in this paper is to investigate the situation when this is the case.

THEOREM 1. *Let* $\{c_j\}_{j=1}^{\infty}$ *be a sequence of non-negative real numbers tending to zero such that* $\sum c_j = \infty$ *and suppose* $F$ *is given as in* (5) *subject to the conditions* (A), (B), *and* (C). *Define* $r: (-1, 1) \backslash \{0\} \to \mathbf{N}$ *by*

$$r(\delta) := \min\{r \in \mathbf{N} : 2^{n_r} |\delta| \geqslant 1\}.$$

*Then*

$$\lim_{\delta \to 0} \left( \sum_{j=1}^{r(\delta)} c_j \right)^{-1} F(x_0 + \delta) = \psi(\theta^*). \tag{7}$$

We note that condition (A) implies that the function $r = r(\delta)$ must satisfy

$$1 \leqslant 2^{n_r} |\delta| \leqslant 2^M \tag{8}$$

whenever $0 < |\delta| < 1$.

*Proof of Theorem* 1. We proceed as in [5] by decomposing the sum defining $F(x)$. Given $0 < |\delta| < 1$,

$$F(x_0 + \delta) = \psi(\theta^*) \sum_{j=1}^{r(\delta)} c_j + S(\delta) + R(\delta)$$

where

$$S(\delta) = \sum_{j=1}^{r(\delta)} c_j [\psi(\theta_j + 2^{n_j}\delta) - \psi(\theta^*)],$$

$$R(\delta) = \sum_{j > r(\delta)} c_j \phi(\theta_j + 2^{n_j}\delta)$$

We claim that for some constant $A_1$ independent of $\delta$,

$$|R(\delta)| \leqslant A_1 \sup_{j \geqslant r(\delta)} c_j, \qquad 0 < |\delta| < 1. \tag{9}$$

To this end, choose $j_0 \in \mathbf{N}$ such that $|\theta_j| \leqslant 2^{j_0 - 1}$, whenever $j \geqslant 1$. It follows that

$$|\theta_j| \leqslant 2^{j - r(\delta) - 1} \leqslant 2^{n_j - n_{r(\delta)} - 1}$$

whenever $j - j_0 \geqslant r(\delta)$, so that for these values of $j$,

$$|\theta_j + 2^{n_j}\delta| \geqslant 2^{n_j - n_{r(\delta)} - 1},$$

where we have used the first inequality in (8). Consequently,

$$\sum_{j > r(\delta) + j_0} (1 + |\theta_j + \delta 2^{n_j}|^N)^{-1} \leqslant 2^N \sum_{j > r(\delta) + j_0} (2^{n_{r(\delta)} - n_j})^N \leqslant 2^N (1 - 2^{-N})^{-1}. \tag{10}$$

Now, we split the summation defining $R(\delta)$ into two: one sum ranging over all $j$ with $r(\delta) + 1 \leqslant j \leqslant r(\delta) + j_0$, and the other ranging over all $j$ with $j > r(\delta) + j_0$. From (10) and (4), it follows that

$$|R(\delta)| \leqslant (j_0 \, \|\psi\| + C 2^N (1 - 2^{-N})^{-1}) \sup_{j > r(\delta)} c_j,$$

where $\|\psi\|$ denotes the maximum of $|\psi(x)|$ as $x$ ranges over **R**. This completes the verification of (9).

On the other hand, an application of Mean Value Theorem immediately yields

$$|S(\delta)| \leqslant \|\psi'\| \cdot (S_1(\delta) + S_2(\delta))$$

where

$$S_1(\delta) = \sum_{j=1}^{r(\delta)} c_j \, |\theta_j - \theta^*|$$

$$S_2(\delta) = |\delta| \sum_{j=1}^{r(\delta)} c_j 2^{n_j}.$$

Since $\sum c_j = \infty$ and $|\theta_j - \theta^*| \to 0$, it follows that

$$\left( \sum_{j=1}^{r(\delta)} c_j \right)^{-1} S_1(\delta) \to 0, \qquad \delta \to 0.$$

In view of this and (9), the proof of Theorem 1 will be complete once we have shown that $S_2(\delta)$ is bounded. Indeed, it is true that

$$\lim_{\delta \to 0} S_2(\delta) = 0.$$

This follows from Lemma 1 below and the simple fact that

$$|\delta| \leqslant 2^{M - n_{r(\delta)}}$$

whenever $0 < |\delta| < 1$. Q.E.D.

LEMMA 1. *Given any increasing sequence $\{p_j\}_{j=1}^{\infty}$ of positive integers and any sequence $\{a_j\}$ of positive numbers tending to zero,*

$$\lim_{r \to \infty} 2^{-p_r} \sum_{j=1}^{r} a_j 2^{p_j} = 0.$$

The proof of this lemma is omitted.

## 3. THE CASE OF MONOTONE COEFFICIENTS

Assuming the convergence of the sequence $\{jn_j^{-1}\}$ and the monotonicity of the coefficients $c_j$ allows for a cleaner reformulation of (7).

THEOREM 2. *Let* $\{c_j\}_{j=1}^{\infty}$ *be a sequence of non-negative real numbers tending monotonically to zero such that* $\sum c_j = \infty$ *and suppose* $F$ *is given as in* (5) *subject to the conditions* (A), (B), (C), *and* (D).

*Define* $s\colon (-1, 1)\backslash\{0\} \to \mathbf{N}$ *by*

$$s(\delta) := [q^*(\log 2)^{-1} \log(|\delta|^{-1})],$$

*where* $q^* = \lim_{j\to\infty} jn_j^{-1}$, *and* $[x]$ *denotes the greatest integer less than or equal to* $x$. *Then*

$$\lim_{\delta\to 0} \left(\sum_{j=1}^{s(\delta)} c_j\right)^{-1} F(x_0+\delta) = \psi(\theta^*). \tag{11}$$

*Proof of Theorem* 2. In view of (7), it is sufficient to show that

$$\sum_{j=1}^{s(\delta)} c_j \simeq \sum_{j=1}^{r(\delta)} c_j, \qquad \delta \to 0,$$

which will follow if we can show that

$$\left(\sum_{k=1}^{m(\delta)} c_k\right)^{-1} \left(\sum_{k=m(\delta)+1}^{M(\delta)} c_k\right) \to 0,$$

where $m(\delta) = \min\{r(\delta), s(\delta)\}$ and $M(\delta) = \max\{r(\delta), s(\delta)\}$.

We observe that by condition (A), $n_k n_{k-1}^{-1} \leqslant 1 + M n_{k-1}^{-1}$, for all integers $k > 1$. Combining this with the definition of $r = r(\delta)$ given in the statement of Theorem 1, one arrives at

$$L(\delta) \leqslant n_r \leqslant (1 + M n_{r-1}^{-1})\, L(\delta)$$

with $L(\delta) = (\log 2)^{-1} \log(|\delta|^{-1})$. This, in conjunction with the hypothesis $\lim r n_r^{-1} = q^*$ shows that $r(\delta) \simeq q^* L(\delta)$, from which follows

$$m(\delta) \simeq M(\delta), \qquad \delta \to 0.$$

Finally, the monotonicity of the sequence $\{c_j\}$ implies

$$\frac{1}{M(\delta) - m(\delta)} \sum_{m(\delta)+1}^{M(\delta)} c_k \leqslant \frac{1}{m(\delta)} \sum_{1}^{m(\delta)} c_k$$

and therefore

$$\left(\sum_{1}^{m(\delta)} c_k\right)^{-1} \sum_{m(\delta)+1}^{M(\delta)} c_k \leqslant M(\delta)\, m(\delta)^{-1} - 1 \to 0.$$

This completes the proof of Theorem 2. Q.E.D.

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